Garbage Collection for a calculus with indices

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Outline

Why study λ-calculus related theories

Invented by A. Church in the '30. Why study it?

- λ-calculus is the base of functional programming and theorem provers.
- Simple and yet powerful.
- But its naïve implementation is not useful in all cases, and there might be no obvious extension for what is needed.
- For the Automath theorem prover, de Bruijn created a way to unique identify variables by the use of *indices*.
- Also, the need for performance lead to *explicit substitutions*.

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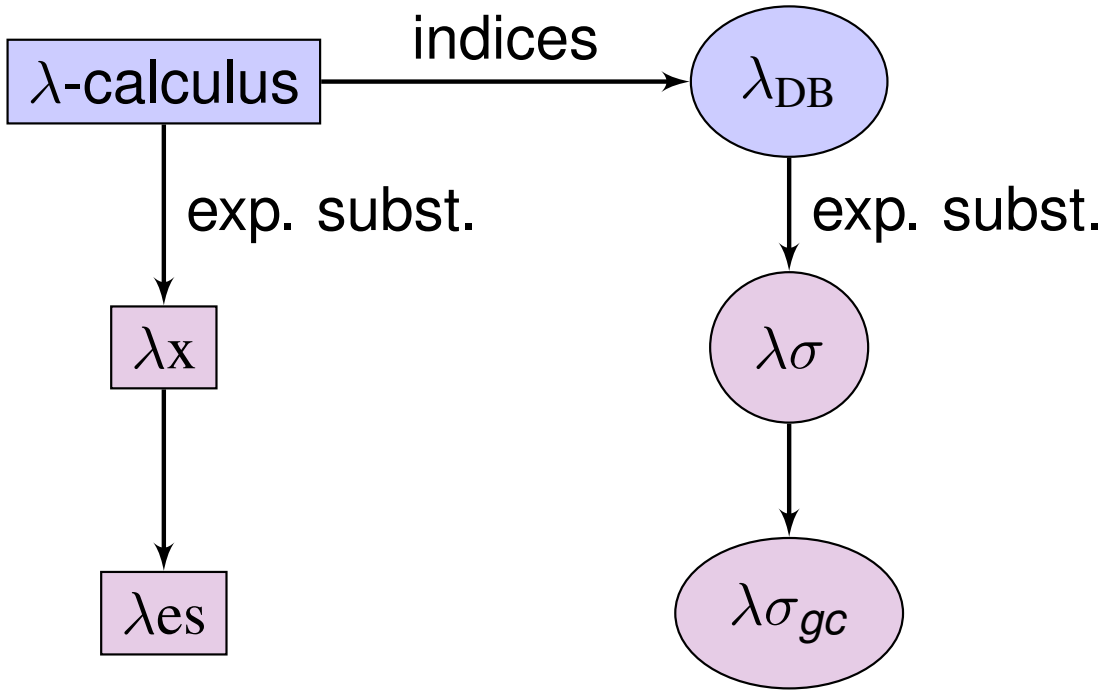
Part of the λ-calculus hierarchy

What we are going to see



Part of the λ-calculus hierarchy

What we are going to see

λ-calculus

The *F* programming language

- The terms of *F* are constructed by
  - Naturals $(0, 1, \ldots)$ with built-in operations $+, \times, \ldots, >, <, =, \ldots$
  - Booleans (*true*, *false*) with built-in operations *if* $\cdot$ *then* $\cdot$ *else* $\cdot, \wedge, \vee, \ldots$
  - Variables $x, y, \ldots$
  - Functions (for instance, $f(x) = x + 1$). In this case x is the *abstracted* variable of f and $x + 1$ is the body.
  - Application of terms to terms (for instance, $f\ 3$). Here, 3 is the *argument* of f. Application is left-associative: $f\ 2\ 1$ is $(f\ 2)\ 1$.
- *One computation step* ($\rightarrow$) is defined to an application $f\ T$ as the act to replace each occurrence of the abstracted variable in the body with the argument. For example, $f\ 3 \rightarrow 3 + 1 = 4$

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Examples

$$fc(x) = \mathit{if}\ (x > 1)\ \mathit{then}\ x \times fc\ (x-1)\ \mathit{else}\ 1$$

$$fc\ 3 \rightarrow \mathit{if}\ (3 > 1)\ \mathit{then}\ 3 \times fc\ (3-1)\ \mathit{else}\ 1 = 3 \times fc\ 2 \rightarrow \ldots$$

$$\rightarrow 3 \times 2 \times 1 = 6$$

$$f(x) = x + 1 \qquad h(y) = y\ 4$$

$$g(x) = x \times 2 \qquad o(x) = x\ x$$

$$h\ f \rightarrow f\ 4 \rightarrow 4 + 1 = 5$$

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From F to λ-calculus

Pure λ-calculus is similar to F, but

- It does not have constants (i.e. naturals and booleans).
  How do we do arithmetic?
- Functions are represented as "inline" anonymous functions.
  - In F the definition and the application of a function are different things:
    Definition: $f(x) = T$. Application: $f\ U$.
  - In λ-calculus both concepts are joined.
    $f\ U$ is written as $(\lambda x.T)\ U$.
    $f\ (f\ U)$ is written as $(\lambda x.T)\ ((\lambda x.T)\ U)$.
    How can we do a recursive call?

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Formalization

The computation step, called β-reduction, is defined as

$$(\lambda x.T)\ U \quad \rightarrow_\beta \quad T[\![x/U]\!]$$

And the substitution is defined as

$$\begin{array}{rcll} x[\![x/T]\!] & = & T & \\ y[\![x/T]\!] & = & y & (x \neq y) \\ (T_1\ T_2)[\![x/U]\!] & = & T_1[\![x/U]\!]\ T_2[\![x/U]\!] & \\ (\lambda y.T)[\![x/U]\!] & = & \lambda y.(T[\![x/U]\!]) & \end{array}$$

We will use the convention that every abstracted variable is different.

An introduction on λ-calculus .

Example.

$$(\lambda x.(\lambda y.x))\ T \rightarrow_\beta (\lambda y.x)[\![x/T]\!] = (\lambda y.T)$$

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α-congruence

Are these two terms different?

$$(\lambda x.x) \qquad\qquad (\lambda y.y)$$

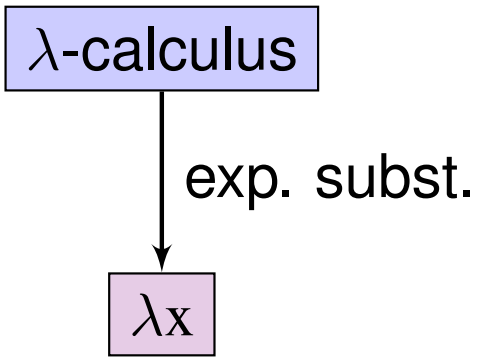
Syntactically, yes, but they behave the same way.

We will not make that distinction, by use of the

α-congruence: Two terms are α-congruent if they only differ in the *bounded variables*.

Part of the λ-calculus hierarchy

What we are going to see



Explicit substitutions

- In traditional λ-calculus the substitution is made all at once.

$$(\lambda x.(\lambda y.x)\ x)T \ \rightarrow_\beta \ ((\lambda y.x)\ x)[\![x/T]\!] = ((\lambda y.T)\ T)$$

- This can lead to a size explosion when having multiple copies of a variable.
- In practice, we need more control over what's going on during the substitution.

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Presenting λx

Presented by K. Rose in 1993, λx is the shortest, simpler calculus with explicit substitutions:

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\text{(B)} & (\lambda x.T)\ U & \longrightarrow & T[x/U] & \\
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Note: The substitution is now part of the terms. We can construct the term $x[y/w]$ and start the computation from there.

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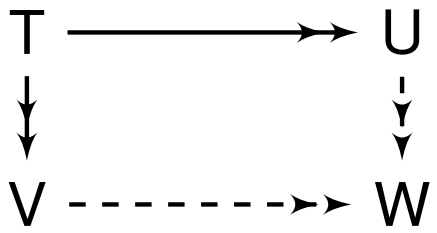
Properties of λx

- λx is *confluent*:



- A term is **strongly normalizing** if there is no infinite computation from it.
  λx *preserves strong normalization*:
  All the terms that are *strongly normalizing* in λ-calculus are also strongly normalizing in λx.

- λx is **not** *metaconfluent*: If we add *metavariables* (X, Y, ...) to the calculus, then confluence is lost.

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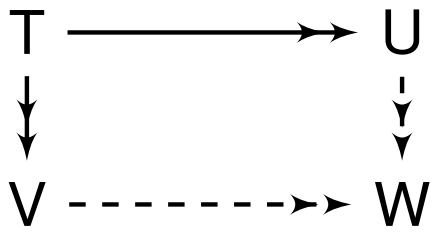

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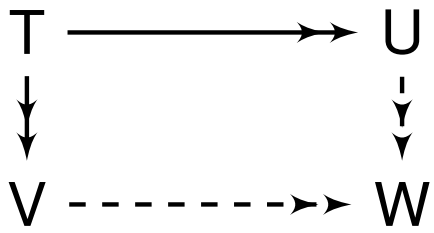

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Part of the λ-calculus hierarchy

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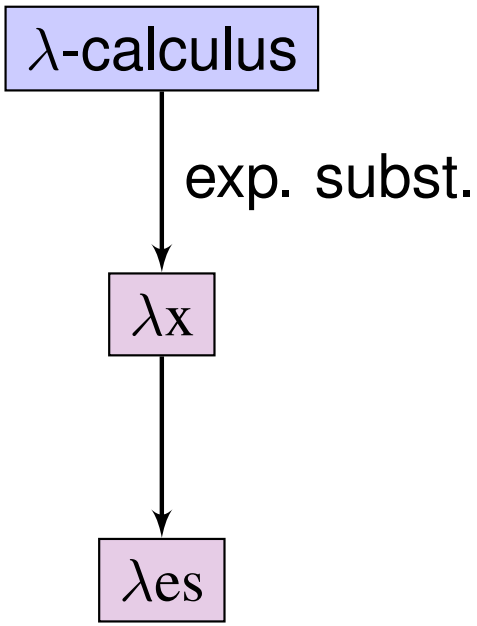

λes essentials

The calculus λes was created in 2007 by Delia Kesner. It gets preservation of strong normalization and metaconfluence by mixing two ideas:

- Composition of substitutions:
  When having two substitutions together, put the second in the first one:

$$T[x/U][y/V] \rightarrow T[y/V][x/U[y/V]]$$

  Needed to get metaconfluence. If no restriction is specified, the preservation of strong normalization is lost.

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λes essentials

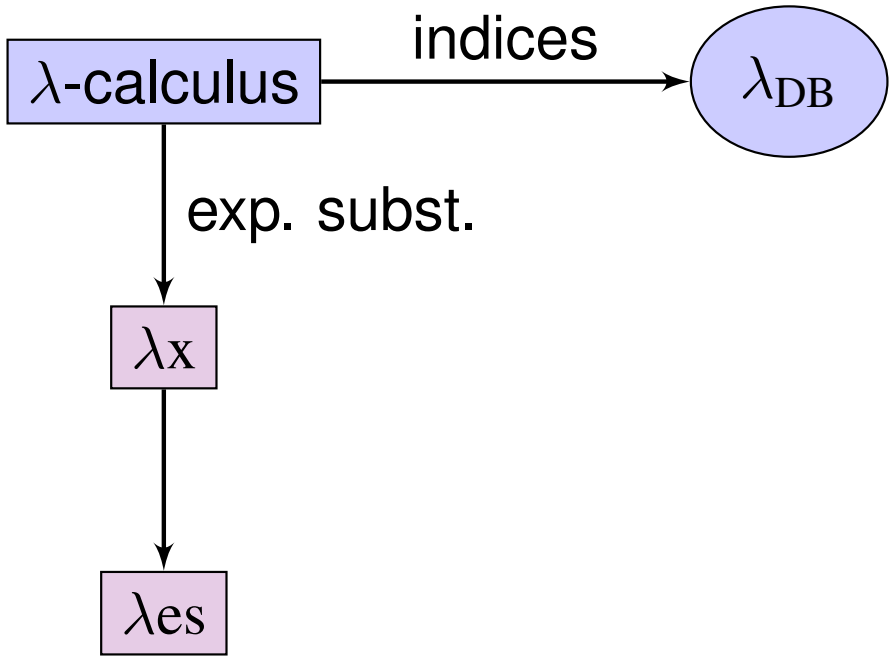
- Garbage collection:
  A substitution $[x/U]$ is *garbage* for the term T if x is not free in T.
  The idea is that T will remain unchanged after performing the substitution, *i. e.*

$$T[x/U] \rightarrow \ldots \rightarrow T$$

  Needed to control the composition, avoiding composition of garbage.

Part of the λ-calculus hierarchy

What we are going to see



Presenting λ_{DB}

Created by N. G. de Bruijn in 1972 for his Automath theorem prover.

- Motivation: knowing whether two terms are "the same" without making variable replacements. Remember the notion of α-congruence:

$$(\lambda x.(\lambda y.x\ y)) \equiv_\alpha (\lambda w.(\lambda z.w\ z))$$

- Idea: for each variable, count how many λ's there are in between, and replace the variable by this number.

$$(\lambda x.(\lambda y.x\ y\ v)) \quad \text{is translated to} \quad (\lambda(\lambda 2\ 1\ 3))$$

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Presenting λ_{DB}

Advantages:

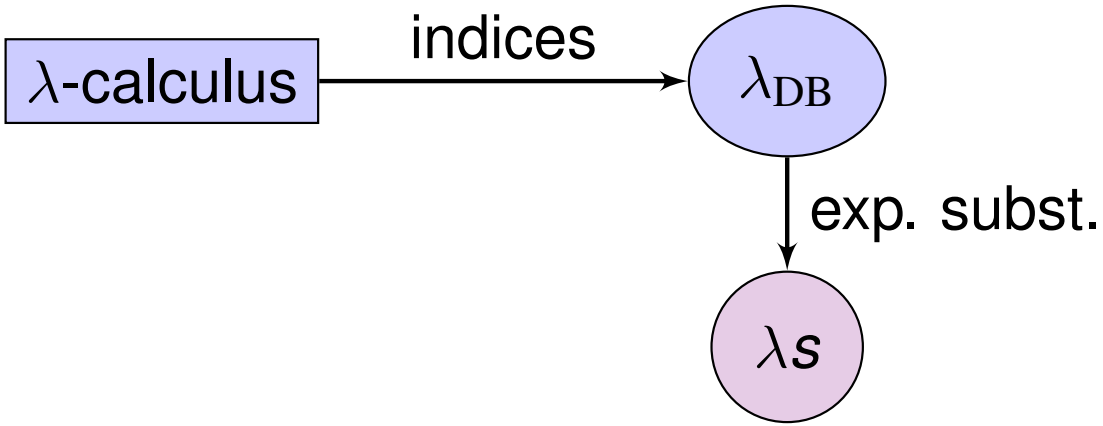
All the terms of the same class are encoded to the same λ_{DB}-term.

Disadvantages:

Indices need to be updated:

- Each time we perform a β-reduction.
- Each time we perform a substitution under a λ.

A word about λs

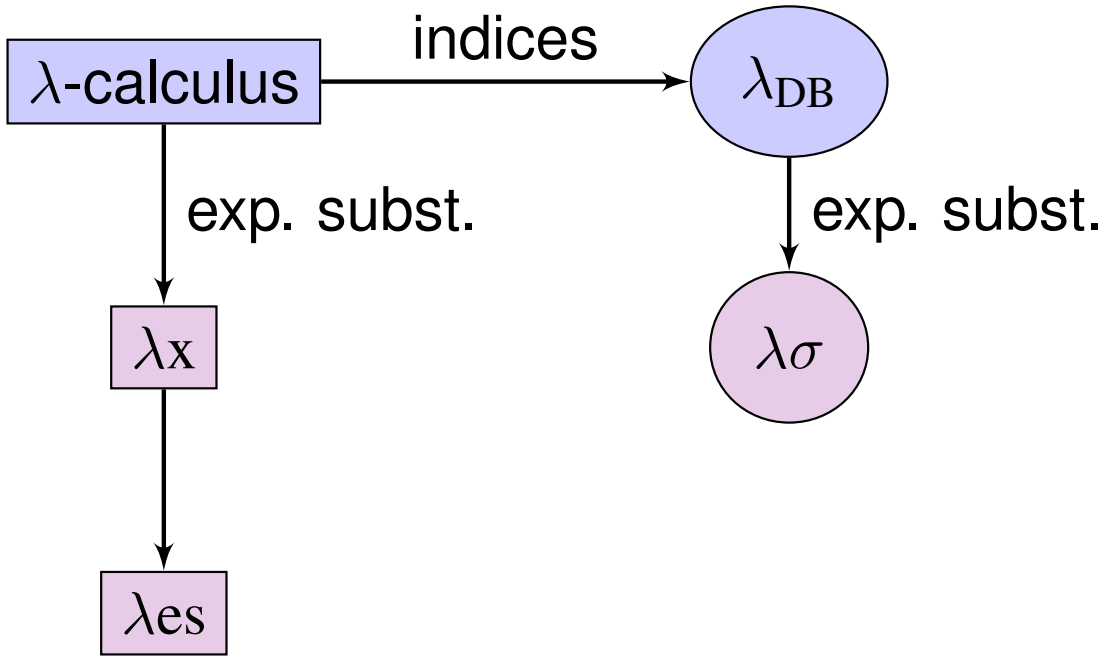


In 1995, F. Kamareddine and A. Ríos proposed a version of λ_{DB} with explicit substitutions, λs.

- As λx, is a direct translation of the rules into exp. sust.
- Holds the same properties as λx.
- Because of the indices updating mechanism, it is hard to extend it with composition of substitutions.

Part of the λ-calculus hierarchy

What we are going to see



Indices and composition of substitutions: $\lambda\sigma$

$\lambda\sigma$ was introduced by Abadi, Cardelli, Curien, and Levy in 1991.
The main features are:

- Multiple substitutions:
  Substitutions are a list $(1/a_1, 2/a_2, \ldots, n/a_n)$
- Born with composition of substitutions in mind.
- Indices updating is made in an atomic way.

But:

- It does not preserve strong normalization.
- It is not metaconfluent (but there are extensions that are, or are semi-metaconfluent).

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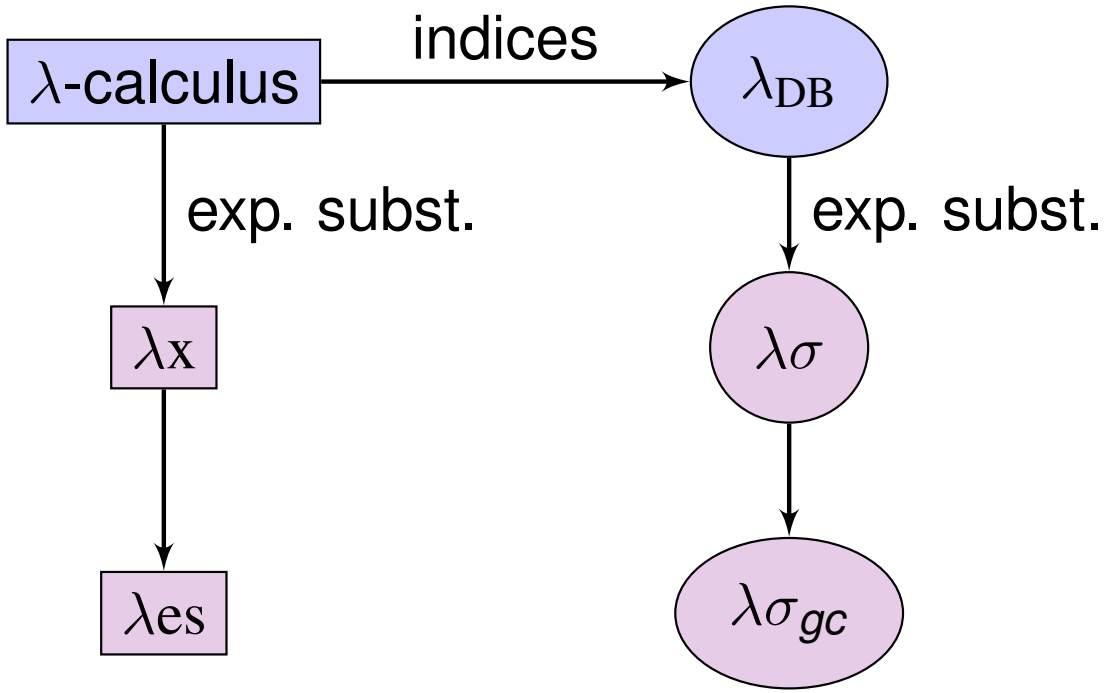

In the search for PSN: $\lambda\sigma_{gc}$

Remember the definition of garbage in λes:
"The substitution is garbage if the variable to substitute does not appear free in the term"

This is difficult to specify in $\lambda\sigma$ for two reasons:

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  The substitution is in charge of the substitution itself **and** the process of index updating, so it is not easy to remove.
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$\lambda\sigma_{gc}$: Results and open problems

What we do and what we know:

- First definition of garbage for $\lambda\sigma$.
- At least, the same good properties as $\lambda\sigma$.
- A known term that is a counterexample for no preservation of strong normalization in $\lambda\sigma$ is strongly normalizing in $\lambda\sigma_{gc}$.
- Is not metaconfluent.
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What we don't know:

- Does it preserve strong normalization?
- Is there a trivial extension to obtain metaconfluence?
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Summary

- There are a lot of calculi related to λ-calculus . Some of them use indices, others explicit substitutions. Some of them use both concepts.
- It is not easy to get preservation of strong normalization and metaconfluence.
- Garbage collection is needed, but for $\lambda\sigma$ it is hard to add .

The end

Questions?

Understunding λ_{DB} by example

$$(\lambda(\lambda 1\ 2\ 3))\ b \rightarrow_\beta (\lambda 1\ 2\ 3)[\![1/b]\!]$$

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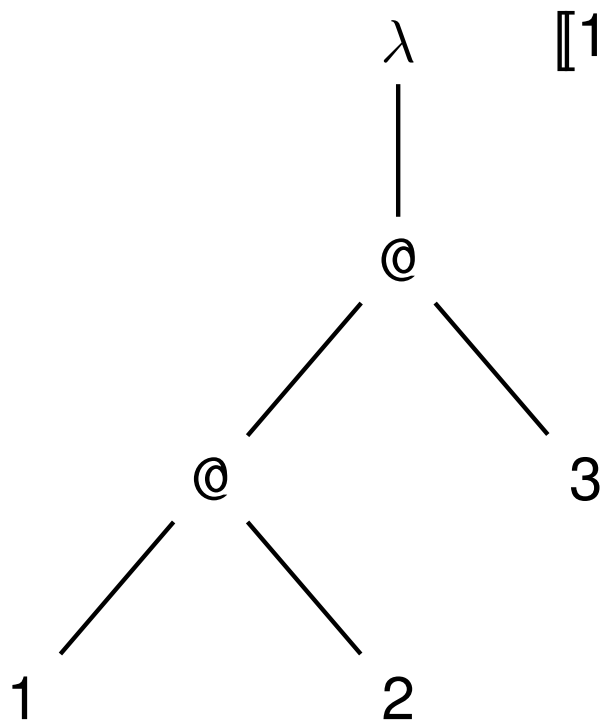

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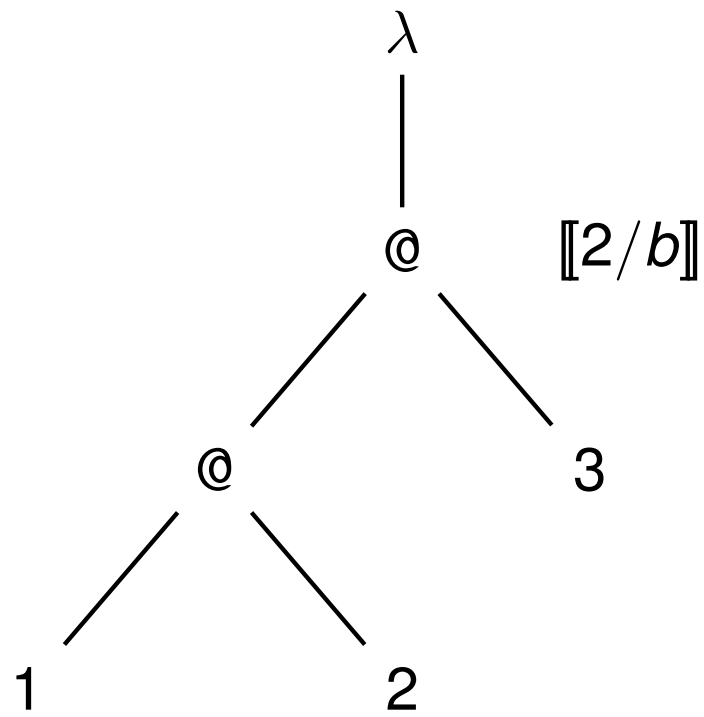

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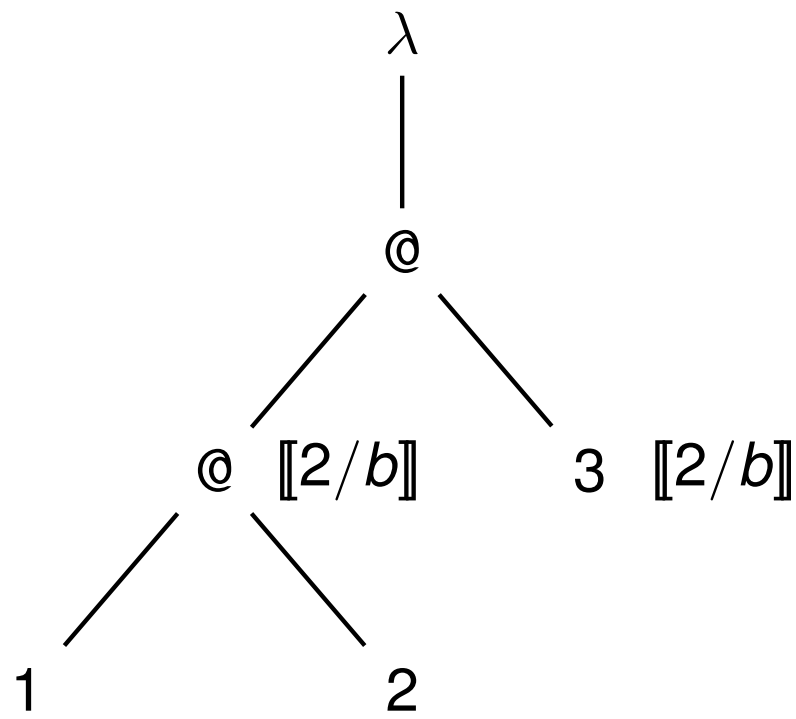

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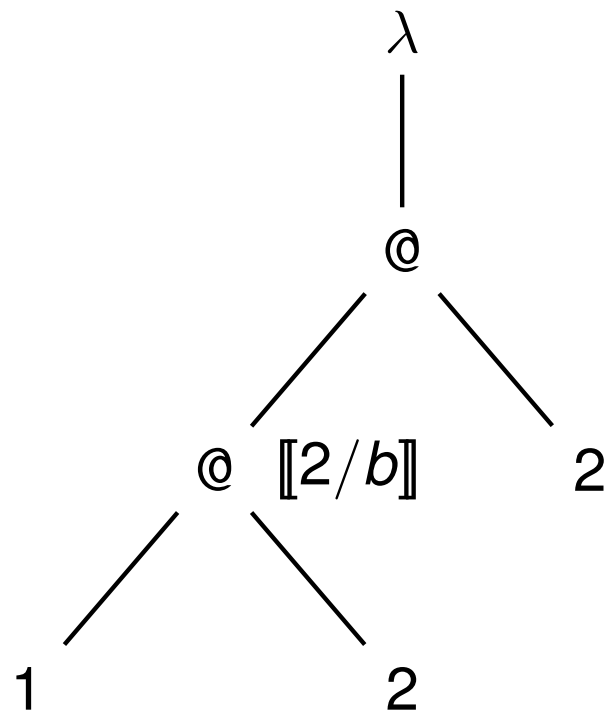

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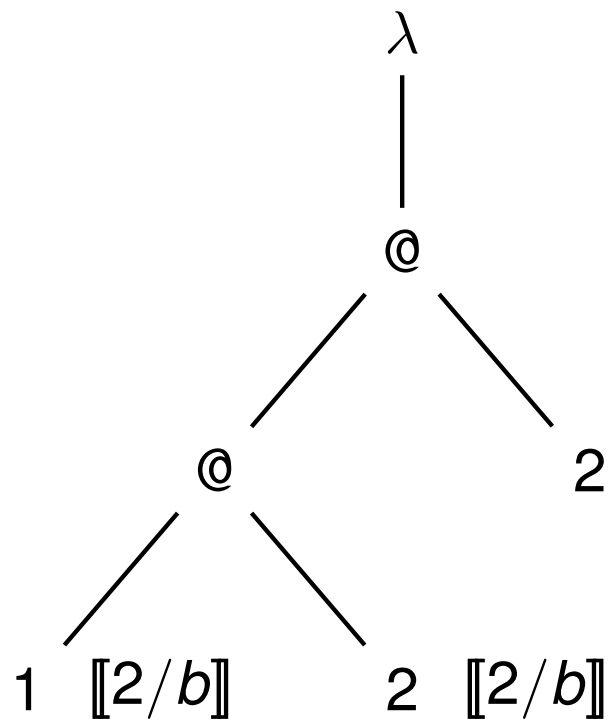

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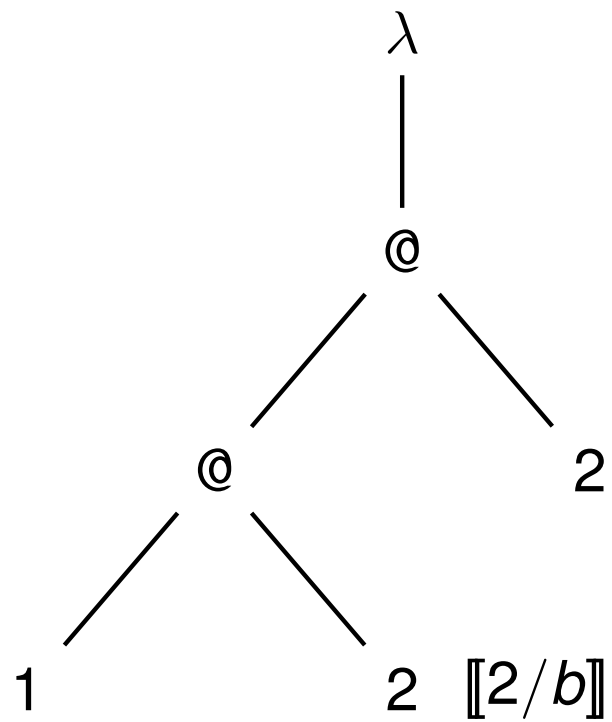

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Understunding λ_{DB} by example

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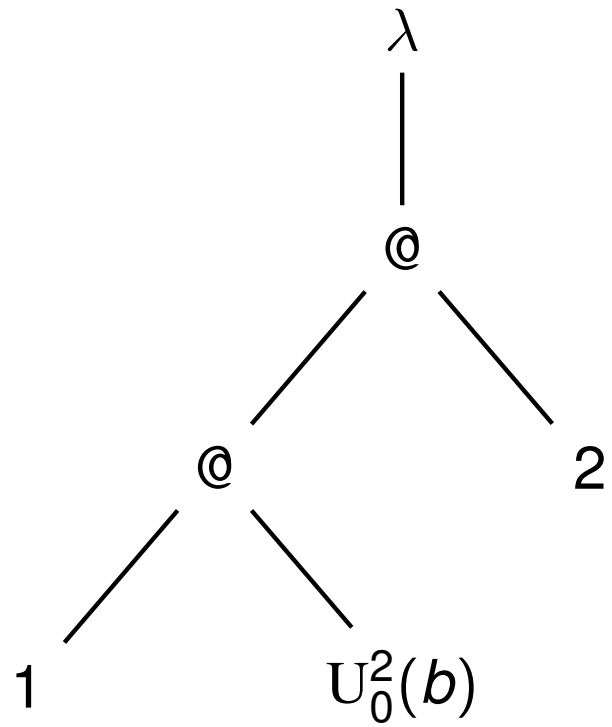

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Understunding λ_{DB}

The update operator

The idea of the update operator $\mathrm{U}_k^i(b)$ is:

- Leave al bounded variables of b untouched.
- Update the free variables by adding $i-1$, the number of λ's crossed.

So

- The parameter i never changes.
- The parameter k is increased each time we cross a λ.